Differentiability of the stable norm in codimension one

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Abstract

The real homology of a compact, n-dimensional Riemannian manifold M is naturally endowed with the stable norm. The stable norm of a homology class is the minimal Riemannian volume of its representatives. If M is orientable the stable norm on $H_{n-1}(M,\mathbb{R})$ is a homogenized version of the Riemannian $(n-1)$-volume. We study the differentiability properties of the stable norm at points $\alpha \in H_{n-1}(M,\mathbb{R})$. They depend on the position of α with respect to the integer lattice $H_{n-1}(M,\mathbb{Z})$ in $H_{n-1}(M,\mathbb{R})$. In particular, we show that the stable norm is differentiable at α if α is totally irrational.

1 Introduction

On every compact Riemannian manifold M the real homology vector spaces $H_m(M,\mathbb{R})$ are endowed with a natural norm, called *stable* or *mass norm*. The stable norm $\mathcal{S}(\alpha)$ of $\alpha \in H_m(M,\mathbb{R})$ is defined as the infimum of the Riemannian m-volumes of real singular cycles representing α. Equivalently, $\mathcal{S}(\alpha)$ can be defined as the minimum of the masses of closed m-currents representing α. The term "stable norm" was coined by M. Gromov, cf. [Gr, Chapter 4]. The concept itself was introduced prior to this by H. Federer [Fe2, 4.15]. If $m = 1$, or if M is n-dimensional and orientable and $m = n-1$, then the stable norm is a homogenized version of the Riemannian length or

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$(n-1)$-volume functional, respectively. Here, homogenization is performed with respect to $\mathbb{Z}^b$ acting as group of deck transformations on the Abelian covering of M, where $b = b_1(M)$ denotes the first Betti number.

We study differentiability properties of the stable norm $\mathcal{S}$ in the codimension one case, i.e. in the case $m = n-1$. At a point $\alpha \in H_{n-1}(M, \mathbb{R})$ the existence of two-sided directional derivatives of $\mathcal{S}$ at α depends on the position of α with respect to the integer lattice $H_{n-1}(M, \mathbb{Z})$ in $H_{n-1}(M, \mathbb{R})$.

1.1 Theorem. *Let M be a compact, orientable, n-dimensional Riemannian manifold and $\mathcal{S}\colon H_{n-1}(M, \mathbb{R}) \to \mathbb{R}_{\geq 0}$ the associated stable norm on $H_{n-1}(M, \mathbb{R})$. If $\alpha \in H_{n-1}(M, \mathbb{R})$, let $V(\alpha)$ denote the smallest linear subspace of $H_{n-1}(M, \mathbb{R})$ that is spanned by integer classes and contains α. Then the restriction of $\mathcal{S}$ to $V(\alpha)$ is differentiable at α.*

The extremal cases are that α is rationally independent, in which case $V(\alpha) = H_{n-1}(M, \mathbb{R})$ and $\mathcal{S}$ is differentiable at α, and the case that the direction of $\alpha \neq 0$ is rational, in which case $\dim V(\alpha) = 1$ and the claim of Theorem 1.1 is obvious.

Due to the convexity and homogeneity of $\mathcal{S}$ the claim of Theorem 1.1 can be stated in the following alternative form.

The tangent cone of the unit ball $B := \left\{\beta \in H_{n-1}(M, \mathbb{R}) \;\middle|\; \mathcal{S}(\beta) \leq 1\right\}$ at $\alpha \in \partial B$ splits as a product, with one factor a hyperplane in $V(\alpha)$.

There is strong evidence that this result is optimal, in the sense that for a large set of Riemannian metrics on an n-torus T^n the stable norm on $H_{n-1}(T^n, \mathbb{R}) \simeq \mathbb{R}^n$ is two-sided differentiable precisely in the directions covered by Theorem 1.1, cf. [Ba1], [Se2], [CdlL, 10.4]. On the other hand, for flat metrics on T^n the stable norm on $H_{n-1}(T^n, \mathbb{R})$ is induced by a scalar product. For this and other explicit examples, see [Fe2, 4.15].

In the case of the 2-torus T^2 the theorem is proved in [Ba1], see also [Au] and [Ma]. For closed, orientable Riemannian surfaces F of genus $s > 1$ the boundary structure of the stable norm ball $B \subseteq H_1(F, \mathbb{R}) \simeq \mathbb{R}^{2s}$ is studied in [Ms1] and [Ms2]. In particular, in this case Theorem 1.1 follows from [Ms2, Corollary 3]. W. Senn [Se1], [Se2], [Se3] proved results analogous to Theorem 1.1 for $\mathbb{Z}^n$-periodic nonparametric variational problems. The case of the stable norm on $H_1(T^n, \mathbb{R})$ for $n > 2$ is considerably more subtle, see [BIK].

Although the basic idea of our proof for Theorem 1.1 is simple, we meet some complications that are caused by the lack of regularity of the objects

involved. Here, we give a rough sketch of the proof that does not attend to such subtleties. The proof is based on the duality between real homology and cohomology realized by flat cycles and cocycles from Geometric Measure Theory. Dual to the stable norm on $H_m(M,\mathbb{R})$ we have the comass norm on the de Rham cohomology vector space $H^m_{\mathrm{dR}}(M)$. Here the comass of a cohomology class $l \in H^m_{\mathrm{dR}}(M)$ is the infimum of the maximum norms of smooth closed m-forms representing l. An important point in the proof is the existence of a bounded, measurable, weakly closed m-form λ that represents l, realizes this infimum and that is defined by a process of differentiation from a flat cocycle representing l. This is due to J. H. Wolfe, cf. [Wh].

Differentiability of $\mathcal{S}|V(\alpha)$ at $\alpha \in H_{n-1}(M,\mathbb{R})$ means that $l_1(\beta) = l_2(\beta)$ whenever $\beta \in V(\alpha)$ and $l_1, l_2 \in H^{n-1}_{\mathrm{dR}}(M)$ are subderivatives of $\mathcal{S}$ at α. Hence, in order to prove Theorem 1.1, we will show that

$$(l_1 - l_2)(\beta) = 0\,. \tag{1}$$

We let λ_1, λ_2 denote the $(n-1)$-forms representing l_1, l_2 mentioned above. We choose a smooth closed 1-form η that represents the Poincaré dual of β, so that

$$(l_1 - l_2)(\beta) = \int_M \eta \wedge (\lambda_1 - \lambda_2)\,.$$

Finally, there exists a closed $(n-1)$-current T representing α and of minimal mass $\mathbf{M}(T) = \mathcal{S}(\alpha)$. Then the assumption that l_1 and l_2 are subderivatives of $\mathcal{S}$ at α can be used to prove that λ_1 and λ_2 coincide at Lebesgue almost all points of $\operatorname{spt} T$. So, in order to prove (1) it suffices to show that

$$\int_{M \smallsetminus \operatorname{spt} T} \eta \wedge (\lambda_1 - \lambda_2) = 0\,.$$

According to [AB1] the current T can be represented as a measured lamination by minimizing hypersurfaces (possibly with singularities). The connected components of $M \smallsetminus \operatorname{spt} T$ are called the gaps of T and it remains to prove that

$$\int_G \eta \wedge (\lambda_1 - \lambda_2) = 0 \tag{2}$$

for every gap G of T. Using the fact that $\beta \in V(\alpha)$, one can see that η is exact on G, $\eta = dg$ for some function $g \in C^\infty(G,\mathbb{R})$. Hence, in the weak sense, we have

$$\eta \wedge (\lambda_1 - \lambda_2) = d\big(g\,(\lambda_1 - \lambda_2)\big) \tag{3}$$

on G. Now, one would like to integrate (3) over G using Stokes's theorem and to conclude that the boundary terms vanish since $\lambda_1 = \lambda_2$ on ∂G. This would prove (2). Actually, one has to be more careful at this point since on the one hand $\overline{G}$ is not a compact domain with smooth boundary, and on the other hand λ_1 and λ_2 need not be defined on ∂G.

In the course of the proof we obtain the following result which may be of independent interest. For the notions in this statement see the beginnings of Sections 3.1 and 4.

4.2 Theorem. *Suppose that the flat cocycle L is a calibration and that the closed rectifiable current $T \in \mathcal{R}_{n-1}(M)$ is calibrated by L. Then the singular set of T is contained in the singular set of L.*

In particular, the union of the singular sets of all closed $T \in \mathcal{R}_{n-1}(M)$ calibrated by L is a Lebesgue null set.

As the preceding statement shows we have to use notions and results from Geometric Measure Theory. In order to make the article reasonably comprehensive we give definitions for most of the concepts that are used in an essential way. These are of a functional analytic nature and easily comprehensible. In Section 2 we treat the existence of mass-minimizing currents in real homology classes. Section 3 develops the dual theory of flat cocycles, calibrations and their representation by weakly closed L^∞-forms. In Section 4 we specialize to the codimension one case and prove Theorem 4.2 above. In Section 5 we summarize results on the structure of mass-minimizing currents in codimension one real homology classes. These are formulated for the lift of the current to the smallest (infinite) covering space on which this lift bounds. Finally, in Section 6, the proof of Theorem 1.1 is completed.

2 Currents and the Stable Norm

2.1 The stable norm

Throughout the paper, M will denote an oriented, n-dimensional Riemannian manifold and m an integer, $0 \le m \le n$. In this subsection we assume in addition that M is compact.

The *stable* (or *mass*) *norm* $\mathcal{S}(\alpha)$ of a real homology class $\alpha \in H_m(M, \mathbb{R})$ is defined as the infimum of the Riemannian volumes of all real Lipschitz cycles $c = \sum r_i \sigma_i$ representing α, see [Gr, 4.C]. Here the volume $\mathrm{vol}_m(c)$

of c is $\sum|r_i| \operatorname{vol}_m(\sigma_i)$, where $\operatorname{vol}_m(\sigma_i)$ denotes the m-dimensional Riemannian volume of the Lipschitz simplex $\sigma_i \colon \Delta^m \to M$. To see that $\mathcal{S}(\alpha) > 0$ if $\alpha \neq 0$, note that there exists a de Rham cohomology class $\beta \in H^m_{\mathrm{dR}}(M)$ such that

$$0 < [\alpha, \beta] = \int_c \omega \leq \operatorname{vol}_m(c) \, \|\omega\|_\infty ,$$

whenever c represents α and the closed m-form ω represents β.

In general, there will not exist a real Lipschitz cycle c in a given homology class $\alpha \in H_m(M, \mathbb{R})$ such that the volume of c equals the stable norm $\mathcal{S}(\alpha)$ of α. Geometric Measure Theory provides an appropriate notion of weak solution to this problem – the normal currents.

2.2 Normal and locally normal currents

In the following we do not assume that M is compact. We consider the chain complex

$$\partial \colon \left(\Omega^{m+1}_0 M\right)^* \to \left(\Omega^m_0 M\right)^*$$

that is dual to the complex

$$d \colon \Omega^m_0 M \to \Omega^{m+1}_0 M$$

given by the spaces $\Omega^m_0 M$ of smooth m-forms ω with compact support $\operatorname{spt} \omega$ in M and the exterior derivative. On $\Omega^m_0 M$ we consider the *comass norm*

$$\|\omega\|_\infty := \max_{x \in M} \|\omega_x\| ,$$

where $\|\omega_x\|$ denotes the (pointwise) *comass norm* of $\omega_x \in \Lambda^m TM_x$,

$$\|\omega_x\| = \max \left\{ \omega_x(e_1, \ldots, e_m) \;\middle|\; e_i \in TM_x \text{ and } |e_i| \leq 1 \text{ for } 1 \leq i \leq m \right\} .$$

The *mass* $\mathbf{M}(T) \in [0, \infty]$ of $T \in \left(\Omega^m_0 M\right)^*$ is defined by

$$\mathbf{M}(T) = \sup \left\{ T(\omega) \;\middle|\; \omega \in \Omega^m_0 M \text{ and } \|\omega\|_\infty \leq 1 \right\} .$$

We say that T is of *locally finite mass* if the open sets $U \subseteq M$ such that

$$\mathbf{M}_U(T) = \sup\left\{ T(\omega) \;\middle|\; \omega \in \Omega^m_0 U \text{ and } \|\omega\|_\infty \leq 1 \right\} < \infty$$

cover M. In this case T is representable by integration, i.e., there exist a positive Radon measure μ_T and a μ_T-measurable unit m-vector field $\vec{T}$ such that

$$T(\omega) = \int \langle \vec{T}, \omega \rangle \, d\mu_T$$

for all $\omega \in \Omega_0^m M$. Clearly $\mu_T(U) = \mathbf{M}_U(T)$ if $U \subseteq M$ is open. If T is of locally finite mass and $f\colon M \to \mathbb{R}$ is locally μ_T-integrable then $T \llcorner f$, defined by

$$\big(T \llcorner f\big)(\omega) = \int f \, \langle \vec{T}, \omega \rangle \, d\mu_T \,,$$

is of locally finite mass. If $A \subseteq M$ is μ_T-measurable one sets

$$T \llcorner A := T \llcorner \chi_A \,.$$

2.1 Definition. A *locally normal current* T *on* M is an element of $\big(\Omega_0^m M\big)^*$ such that both T and ∂T have locally finite mass. We let $\mathbf{N}_m^{\mathrm{loc}}(M)$ denote the set of locally normal currents on M. A *normal current* is a locally normal current T whose support $\operatorname{spt} T$ $(= \operatorname{spt} \mu_T)$ is compact. We let $\mathbf{N}_m(M)$ denote the set of normal currents.

Note that the mass $\mathbf{M}$ is a norm on the $\mathbb{R}$-vector space $\mathbf{N}_m(M)$. Here are the most important classes of examples of (locally) normal currents.

1. *Lipschitz chains.* If $c = \sum r_i \sigma_i$ is a real Lipschitz m-chain in M, then $[\![c]\!] \in \mathbf{N}_m(M)$ is defined by

   $$[\![c]\!](\omega) = \int_c \omega = \sum r_i \int_{\Delta_m} \sigma_i^* \omega \,.$$

   Stokes's Theorem implies $\partial [\![c]\!] = [\![\partial c]\!]$. Moreover

   $$\mathbf{M}\big([\![c]\!]\big) \leq \mathrm{vol}_m(c) \,.$$

   Contrary to what is stated in [Gr, 4.17], inequality can occur. If parts of the singular simplices cover each other with different orientations, these parts add to $\mathrm{vol}_m(c)$, while they cancel in $[\![c]\!]$. If $m < n$ this situation does not occur for generic chains.

2. *Smooth currents.* If $\eta \in \Omega^{n-m} M$ is a smooth $(n-m)$-form then $T_\eta = [\![M]\!] \llcorner \eta \in \mathbf{N}_m^{\mathrm{loc}}(M)$ is defined by

   $$T_\eta(\omega) = \int_M \eta \wedge \omega \,.$$

   Then one has

   $$\partial T_\eta = (-1)^{n-m+1} T_{d\eta} \,.$$

If $m \in \{0, 1, n-1, n\}$ then

$$\mathbf{M}(T_\eta) = \int_M \|\eta(x)\| \, d\,\mathrm{vol}_n(x)\,.$$

A similar equality is also true for $1 < m < n-1$ if one uses an appropriate norm on $(n-m)$-covectors (namely the norm dual to the comass norm on m-vectors with respect to the duality induced by the wedge product and the volume form). Obviously, one has $T_\eta \in \mathbf{N}_m(M)$ if and only if $\eta \in \Omega_0^{n-m} M$.

2.3 Flat norm

With respect to the mass norm neither the subspace of smooth currents with compact support nor the subspace of Lipschitz chains is dense in $\mathbf{N}_m(M)$. However, this is true with respect to the flat norm $\mathbf{F}$, which is weaker than the mass norm $\mathbf{M}$.

2.2 Definition. The *flat norm* $\mathbf{F}(T)$ of $T \in \mathbf{N}_m(M)$ is defined by

$$\mathbf{F}(T) = \inf \left\{ \mathbf{M}(T + \partial S) + \mathbf{M}(S) \;\middle|\; S \in \mathbf{N}_{m+1}(M) \right\}.$$

One can show that

$$\mathbf{F}(T) = \sup \left\{ T(\omega) \;\middle|\; \omega \in \Omega_0^m M,\ \|\omega\|_\infty \leq 1,\ \|d\omega\|_\infty \leq 1 \right\},$$

cf. [Fe1, 4.1.12], and that the two subspaces mentioned above are $\mathbf{F}$-dense in $\mathbf{N}_m(M)$, cf. [Fe1, 4.1.18 and 4.1.23].

2.4 Homologically mass-minimizing currents

2.3 Definition. A current $T \in \mathbf{N}_m^{\mathrm{loc}}(M)$ is *homologically (mass-)minimizing* if

$$\mathbf{M}_U(T) \leq \mathbf{M}_U(T + \partial S)$$

whenever $U \subseteq M$ is open and relatively compact and $S \in \mathbf{N}_{m+1}(M)$ has support in U.

For the rest of this subsection we additionally assume that M is compact. We approach the question of existence of a mass-minimizing current in every

real homology class. Using de Rham's Theorem, the Hahn–Banach Theorem and the fact that real Lipschitz m-chains are $\mathbf{F}$-dense in $\mathbf{N}_m(M)$, cf. Section 2.3, one can conclude that the homology $H_m(M,\mathbb{R})$ of the chain complex $\partial\colon \mathbf{N}_{m+1}(M) \to \mathbf{N}_m(M)$ is dual to the de Rham cohomology $H^m_{\mathrm{dR}}(M)$. If $\alpha = [T] \in H_m(M,\mathbb{R})$ is represented by the closed current $T \in \mathbf{N}_m(M)$ and $\beta = [\omega] \in H^m_{\mathrm{dR}}(M)$ is represented by the closed form $\omega \in \Omega^m M$, then the natural pairing $H_m(M,\mathbb{R}) \times H^m_{\mathrm{dR}}(M) \to \mathbb{R}$ is given by

$$[\alpha, \beta] = T(\omega)\,.$$

2.4 Proposition. *For every $\alpha \in H_m(M,\mathbb{R})$ one has*

$$\mathcal{S}(\alpha) = \min \big\{ \mathbf{M}(T) \;\big|\; T \in \mathbf{N}_m(M),\ \partial T = 0 \text{ and } [T] = \alpha \big\}\,.$$

Remark. In the framework of Geometric Measure Theory it is natural to take the right hand side of the preceding equation as definition of the stable norm of α. We chose the definition of $\mathcal{S}(\alpha)$ as infimum of the volumes of Lipschitz cycles representing α, since it is geometrically intuitive.

Proof. By [Fe2, 3.9] the minimum on the right hand side is attained. Denote, for the moment, the quantity on the right hand side by $\mathcal{S}'(\alpha)$. By Example 1 above, clearly $\mathcal{S}'(\alpha) \le \mathcal{S}(\alpha)$ for every $\alpha \in H_m(M,\mathbb{R})$.

For the converse, note that since $\mathcal{S}$ and $\mathcal{S}'$ are both norms on $H_m(M,\mathbb{R})$, it suffices to show that they coincide on the dense subset $H_m(M,\mathbb{Q})$. Let $\alpha \in H_m(M,\mathbb{Q})$, $T \in \mathbf{N}_m(M)$ with $\partial T = 0$ and $[T] = \alpha$, and let $\varepsilon > 0$. We suppose that M is isometrically embedded into some $\mathbb{R}^N$. Choose a tubular neighborhood U of M in $\mathbb{R}^N$ so small that the nearest point projection $p\colon U \to M$ satisfies $\mathrm{Lip}(p)^m \le \min\big\{1 + \varepsilon/\mathbf{M}(T), 2\big\}$. By [Fe2, 5.8] there is a closed rectifiable current $R \in \mathcal{R}_m(M)$ (in fact an integral Lipschitz chain) and $k \in \mathbb{N}$ such that $\frac{1}{k}R$ is homologous to T and $\mathbf{M}\big(\frac{1}{k}R\big) \le \mathbf{M}(T) + \varepsilon$. (For the notion of rectifiable current see the beginning of Section 4.)

By [Fe1, Lemma 4.2.19] there exists a closed integral polyhedral chain $P \in \mathcal{P}_m(U)$ (i.e. a linear combination of affine simplices with integer coefficients) and a C^1-diffeomorphism f of U such that $f_\# P$ is homologous to R in U and $\mathbf{M}(f_\# P - R) \le \varepsilon$. After a suitable simplicial subdivision we can write P in the form $P = \sum_{i=1}^{l} n_i [\![\Delta_i]\!]$ where the Δ_i are affine simplices belonging to a simplicial complex.

Therefore and since f is a diffeomorphism, when calculating the mass there is no cancelation. Putting $\tilde{c} := \sum \frac{n_i}{k}\tilde{\sigma}_i$, where $\tilde{\sigma}_i = f|\Delta_i$, we get $\mathrm{vol}_m(\tilde{c}) = \frac{1}{k}\mathbf{M}(f_\# P) \le \mathbf{M}(\frac{1}{k}R) + \varepsilon \le \mathbf{M}(T) + 2\varepsilon$.

Projecting $\tilde{c}$ to M, we get the desired Lipschitz cycle c representing α and satisfying

$$\mathrm{vol}_m(c) \le \mathrm{Lip}(p)^m \, \mathrm{vol}_m(\tilde{c}) \le \big(1 + \varepsilon/\mathbf{M}(T)\big)\mathbf{M}(T) + 4\varepsilon \le \mathbf{M}(T) + 5\varepsilon \,.$$

$\square$

Note that a closed current $T \in \mathbf{N}_m(M)$ with $[T] = \alpha$ is homologically mass-minimizing if and only if $\mathbf{M}(T) = \mathcal{S}(\alpha)$. Such a current is called a *(mass-) minimizing current in* α. For $m = 1$ and $m = n - 1$ the structure of homologically mass-minimizing closed currents is well understood, see [Ba2] for the case $m = 1$ and [AB1], [AB2] and Section 5 for the case $m = n - 1$.

The norm on $H^m_{\mathrm{dR}}(M)$ induced by the comass norm on $\Omega^m M$ is equally called *comass norm*, and will be denoted by

$$\mathcal{S}^*(\beta) := \inf \big\{ \|\omega\|_\infty \;\big|\; \omega \in \Omega^m M,\ d\omega = 0 \text{ and } [\omega] = \beta \big\} \,.$$

It is known that the comass norm on $H^m_{\mathrm{dR}}(M)$ is dual to the stable norm $\mathcal{S}$ on $H_m(M, \mathbb{R})$, i.e., for all $\alpha \in H_m(M, \mathbb{R})$ we have

$$\mathcal{S}(\alpha) = \sup \big\{ [\alpha, \beta] \;\big|\; \beta \in H^m_{\mathrm{dR}}(M), \mathcal{S}^*(\beta) \le 1 \big\} \,, \tag{4}$$

see [Fe2, 4.10] or [Gr, 4.35]. Since we need the arguments from the proof of this duality we will reprove it in Section 3.4, Theorem 3.8.

3 Subderivatives of the Stable Norm and Calibrations

3.1 Calibrations

A *flat m-cochain* is a linear functional on $\mathbf{N}_m(M)$ that is continuous with respect to the flat norm $\mathbf{F}$. It is called a *flat cocycle* if it vanishes on the space of boundaries

$$\mathbf{B}_m(M) = \big\{ T \in \mathbf{N}_m(M) \;\big|\; \exists S \in \mathbf{N}_{m+1}(M) \colon \partial S = T \big\}$$

Note that for a flat cocycle L its flat norm

$$\mathbf{F}(L) = \sup \big\{ L(T) \;\big|\; T \in \mathbf{N}_m(M), \mathbf{F}(T) \le 1 \big\}$$

coincides with its comass norm

$$\mathbf{M}(L) = \sup\left\{L(T) \;\middle|\; T \in \mathbf{N}_m(M), \mathbf{M}(T) \leq 1\right\}.$$

Every flat m-cochain can naturally be extended to the $\mathbf{F}$-closure $\mathbf{F}_m(M)$ of $\mathbf{N}_m(M)$, the space of m-dimensional *flat chains*. These flat chains appear in Lemma 3.2, where they are needed in the course of the proof, even if one restricts the statements to the case of normal currents. They will be avoided in the rest of the paper.

3.1 Definition. A flat cocycle L of norm $\mathbf{F}(L) = 1$ is called a *calibration*. If L is a calibration and $T \in \mathbf{F}_m(M)$ satisfies $L(T) = \mathbf{M}(T)$, then L is said to *calibrate* T.

If $T \in \mathbf{N}_m(M)$ is calibrated by a calibration L then T is homologically minimizing. Indeed, if $S \in \mathbf{N}_{m+1}(M)$ then

$$\mathbf{M}(T) = L(T) = L(T + \partial S) \leq \mathbf{M}(T + \partial S).$$

3.2 Lemma. *Let L be a calibration.*

(a) If $(T_i)_{i\in\mathbb{N}}$ is a sequence in $\mathbf{F}_m(M)$ that $\mathbf{F}$-converges to $T \in \mathbf{F}_m(M)$ and if L calibrates each T_i, then L calibrates T.

(b) If $S \in \mathbf{F}_m(M)$ is a piece of $T \in \mathbf{F}_m(M)$, i.e. if $\mathbf{M}(T) = \mathbf{M}(S)+\mathbf{M}(T-S)$, then L calibrates T if and only if L calibrates S and $T - S$.

(c) Assume L calibrates $T \in \mathbf{F}_m(M)$. If $g\colon M \to \mathbb{R}_{\geq 0}$ is μ_T-integrable then L calibrates $T \llcorner g$ and $L(T \llcorner g) = \mathbf{M}(T \llcorner g) = \int g \, d\mu_T$. If $h \in L^1(M, \mu_T)$ then $L(T \llcorner h) = \int h \, d\mu_T$.

Proof. Since the mass is lower semicontinuous with respect to flat convergence we have

$$\mathbf{M}(T) \leq \liminf_{i\to\infty} \mathbf{M}(T_i) = \lim_{i\to\infty} L(T_i) = L(T).$$

This proves (a). Statement (b) follows directly from the definitions. This proves also (c) for the case of step functions. Using (a) and approximation by step functions one obtains (c). (Note that the currents $T \llcorner g_n$, where the g_n are step functions approximating g, need not belong to $\mathbf{N}_m(M)$ even if $T \llcorner g$ does.) □

3.3 Definition. A locally normal m-current $T \in \mathbf{N}_m^{\mathrm{loc}}(M)$ *is calibrated* by the calibration L if $T \llcorner A$ is calibrated by L for every compact set A such that $T \llcorner A \in \mathbf{N}_m(M)$.

3.2 Subderivatives of the stable norm

In this subsection we assume that M is compact. A subderivative of the stable norm $\mathcal{S}\colon H_m(M,\mathbb{R}) \to \mathbb{R}$ at $\alpha \in H_m(M,\mathbb{R})$ is a linear form $l \in H_m(M,\mathbb{R})^*$ such that $l(\alpha) = \mathcal{S}(\alpha)$ and $l(\beta) \le \mathcal{S}(\beta)$ for all $\beta \in H_m(M,\mathbb{R})$.

3.4 Lemma. *Let $l \in H_m(M,\mathbb{R})^*$ be a subderivative of $\mathcal{S}$ at $\alpha \in H_m(M,\mathbb{R})$. Then there exists a calibration $L \in \mathbf{N}_m(M)^*$ such that $L(T) = l\big([T]\big)$ for every closed $T \in \mathbf{N}_m(M)$. In particular, such an L calibrates every minimizing $T \in \mathbf{N}_m(M)$ in the homology class α.*

Remark. Since $\mathcal{S}$ is convex there exists a subderivative l of $\mathcal{S}$ at α for every $\alpha \in H_m(M,\mathbb{R})$. Hence Lemma 3.4 implies that every closed $T \in \mathbf{N}_m(M)$ that minimizes mass in $[T]$ is calibrated by some calibration L.

Proof. For closed currents $T \in \mathbf{N}_m(M)$ we define $L(T) = l\big([T]\big)$. Since $L(T) = l\big([T]\big) \le \mathcal{S}\big([T]\big) \le \mathbf{M}(T)$, we can use the Hahn–Banach Theorem to extend L to a linear functional on all of $\mathbf{N}_m(M)$ such that $\mathbf{M}(L) \le 1$. Since L vanishes on $\mathbf{B}_m(M)$, L is indeed a calibration. If $T \in \mathbf{N}_m(M)$ is closed, $[T] = \alpha$ and $\mathbf{M}(T) = \mathcal{S}(\alpha)$, then

$$L(T) = l(\alpha) = \mathcal{S}(\alpha) = \mathbf{M}(T)\,,$$

i.e., L calibrates T. □

3.3 The canonical representative of a flat cochain

According to their definition flat m-cochains are objects purely from functional analysis. They are elements of $\mathbf{N}_m(M)^*$ that are continuous with respect to the flat norm. But it is well known that flat m-cochains can be represented by bounded Lebesgue measurable m-forms in the following sense, cf. [Wh, IX, Theorem 5A] or [Fe1, 4.1.19].

If L is a flat m-cochain then, by [Wh, IX, Theorem 5A] or [Fe1, 4.1.19] there exists a bounded Lebesgue-measurable m-form λ such that for every smooth m-current T_η, $\eta \in \Omega_0^{n-m}M$, we have

$$L(T_\eta) = \int_M \eta \wedge \lambda\,. \tag{5}$$

We say that λ *is a representative of* L or that λ *represents* L.

If L is closed, i.e. a cocycle, then λ is weakly closed, i.e., $\int_M d\theta \wedge \lambda = 0$ for every $\theta \in \Omega^{n-m-1}M$, and we have $\mathbf{F}(L) = \operatorname{ess\,sup}_{x\in M} \|\lambda(x)\|$, where $\|\ \|$ denotes the comass norm (which coincides with the Euclidean norm if $m = 1$ or $m = n-1$), cf. Section 2.2.

For the proof of Theorem 4.2 it will be important that we can find a canonical representative of L, denoted by D_L, by a process of differentiation. For the following discussion, based on Whitney's book [Wh], we work in standard Euclidean space $\mathbb{R}^n$. Using charts we can apply the results to manifolds (see below).

Given an oriented m-dimensional affine simplex σ, we denote by $P(\sigma)$ the oriented m-dimensional affine subspace of $\mathbb{R}^n$ that contains σ, by $\xi(\sigma) \in \Lambda_m\mathbb{R}^n$ the unit m-vector orienting $P(\sigma)$, and by $\operatorname{vol}(\sigma) = \mathbf{M}\big([\![\sigma]\!]\big)$ the m-dimensional Euclidean volume of σ.

The *thickness* (or *fullness*) $\Theta(\sigma)$ of σ is defined by

$$\Theta(\sigma) = \frac{\operatorname{vol}(\sigma)}{\operatorname{diam}(\sigma)^m}\,.$$

If $p \in \mathbb{R}^n$, then the *p-thickness* $\Theta_p(\sigma)$ of σ is defined by

$$\Theta_p(\sigma) = \frac{\operatorname{vol}(\sigma)}{\operatorname{diam}(\sigma \cup \{p\})^m}\,.$$

The following definition is implicit in [Wh, IX, § 4].

3.5 Definition. Let L be a flat cochain of degree m on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. We say that p is a *regular point* for L if there exists an m-covector $\varphi \in \Lambda^m\mathbb{R}^n$ such that for every m-vector $\xi \in \Lambda_m\mathbb{R}^n$ and all $\varepsilon, \eta > 0$ there exists $\delta > 0$ with the property that every m-simplex $\sigma \subseteq \mathbb{R}^n$ with $p \in \sigma$, $\xi(\sigma) = \xi$, $\operatorname{diam}(\sigma) < \delta$, and $\Theta(\sigma) \geq \eta$ satisfies

$$\left| \langle \xi, \varphi \rangle - \frac{L\big([\![\sigma]\!]\big)}{\operatorname{vol}(\sigma)} \right| < \varepsilon\,.$$

Clearly in this case the covector φ is unique. We denote it by $D_L(p)$.

The set of all regular points for L is denoted by $\operatorname{reg} L$, its complement, the set of all *singular points*, by $\operatorname{sing} L$.

By [Wh, IX, Theorem 5A], $\operatorname{sing} L$ is a Lebesgue null set and the function $D_L \colon \mathbb{R}^n \smallsetminus \operatorname{sing} L \to \Lambda^m\mathbb{R}^n$ is measurable. Moreover, $\|D_L(p)\| \leq \mathbf{F}(L)$ and the bounded m-form $\lambda = D_L$ represents L in the sense of (5)

By [Wh, X, Theorem 9A], regularity is invariant under local diffeomorphisms and $D_L(p)$ behaves like an m-covector. Hence, using local charts, $\operatorname{reg} L$ and D_L are also defined for flat cochains L on a Riemannian manifold.

Proposition 3.6 below proves that $D_L(p)$ can be obtained by a local blow-up of L at p.

For $p \in \mathbb{R}^n$ and $r \geq 0$ we denote by $\mu_{r,p}\colon \mathbb{R}^n \to \mathbb{R}^n$ the homothety with center p and factor r, i.e., $\mu_{r,p}(p+x) = p + rx$.

For $L \in \mathbf{N}_m(\mathbb{R}^n)^*$ a flat m-cochain, $p \in \mathbb{R}^n$, and $r > 0$, we set $L_{r,p} := \frac{1}{r^m}(\mu_{r,p})^\# L$, i.e., for every $T \in \mathbf{N}_m(\mathbb{R}^n)$ we have $L_{r,p}(T) = \frac{1}{r^m} L\big((\mu_{r,p})_\# T\big)$. It is easy to see that if L is represented by the m-form λ then $L_{r,p}$ is represented by $\lambda_{r,p}$ where $\lambda_{r,p}(p+x) = \lambda(p+rx)$.

If $p \in \operatorname{reg} L$, let $L_{0,p}$ denote the flat m-cochain represented by the constant m-form $\lambda_{0,p}$, where $\lambda_{0,p}(x) := D_L(p)$.

3.6 Proposition. *Let L be a flat m-cochain on $\mathbb{R}^n$.*

(a) If $p \in \mathbb{R}^n$ and $0 < r \leq 1$ then $\mathbf{F}(L_{r,p}) \leq \mathbf{F}(L)$.

(b) If $p \in \operatorname{reg} L$ then, for $r \to 0$, $L_{r,p}$ converges to $L_{0,p}$ in the weak-$*$-topology on $\mathbf{N}_m(\mathbb{R}^n)^*$, i.e., $\lim_{r\to 0} L_{r,p}(T) = L_{0,p}(T)$ for every $T \in \mathbf{N}_m(\mathbb{R}^n)$.

Proof. We may assume that $p = 0$. In the following we omit the subscript p. By [Fe1, 4.1.14], for $0 \leq r \leq 1$ and every $T \in \mathbf{N}_m(\mathbb{R}^n)$ we have

$$\mathbf{F}\big((\mu_r)_\# T\big) \leq \max\{r^m, r^{m+1}\}\mathbf{F}(T) \leq r^m \mathbf{F}(T).$$

Therefore,

$$\big|L_r(T)\big| = \frac{1}{r^m}\Big|L\big((\mu_r)_\# T\big)\Big| \leq \frac{1}{r^m}\mathbf{F}(L)\,\mathbf{F}\big((\mu_r)_\# T\big) \leq \mathbf{F}(L)\,\mathbf{F}(T)\,.$$

This proves (a). For the proof of (b), first assume $T = [\![\sigma]\!]$ is given by an oriented m-simplex σ. Denoting $\sigma_r := \mu_r \sigma$ we have

$$L_r(T) = \frac{1}{r^m} L\big([\![\sigma_r]\!]\big) = \operatorname{vol}(\sigma)\frac{L\big([\![\sigma_r]\!]\big)}{\operatorname{vol}(\sigma_r)}$$

Since $\Theta_p(\sigma_r) = \Theta_p(\sigma)$ for every r, [Wh, IX, Theorem 10A] implies

$$\lim_{r\to 0}\frac{L\big([\![\sigma_r]\!]\big)}{\operatorname{vol}(\sigma_r)} = \big\langle \xi(\sigma), D_L(p)\big\rangle$$

and thus

$$\lim_{r\to 0} L_r(T) = \mathrm{vol}(\sigma)\,\big\langle \xi(\sigma), D_L(p)\big\rangle = L_0(T)\,.$$

The fact that polyhedral currents are $\mathbf{F}$-dense in $\mathbf{N}_m(\mathbb{R}^n)$ and (a) imply that $\lim\limits_{r\to 0} L_r(T) = L_0(T)$ for every $T \in \mathbf{N}_m(\mathbb{R}^n)$. $\square$

3.4 Smoothing flat cochains

In the proof of Theorem 1.1 we have to pass from the merely bounded and measurable form λ representing the flat cochain L to a smooth approximation, obtained by convolution. The following lemma is formulated and proved for the case of $M = \mathbb{R}^n$ and the usual convolution in $\mathbb{R}^n$. In the general case, one can embed M into some $\mathbb{R}^N$ and perform the convolution using a tubular neighborhood of the submanifold M. See also [Fe2, 4.7].

3.7 Lemma. *Suppose $L \in \mathbf{N}_m(\mathbb{R}^n)^*$ is a flat cochain, represented by the bounded, measurable m-form λ. For $\varepsilon > 0$, let λ_ε originate from λ by convolution with the kernel φ_ε.*

Then we have

$$\lim_{\varepsilon\to 0} T(\lambda_\varepsilon) = L(T)$$

for every $T \in \mathbf{N}_m(\mathbb{R}^n)$.

Remark. Note that the statement is standard if λ is a smooth form or if T is given by a smooth form.

Proof. The mollified currents T_ε are defined by $T_\varepsilon(\omega) = T(\omega_\varepsilon)$ for every $\omega \in \Omega_0^m\mathbb{R}^n$. T_ε is a smooth current, $T_\varepsilon = T_{\eta_\varepsilon}$ for some $\eta_\varepsilon \in \Omega_0^{n-m}\mathbb{R}^n$. In particular, $T_\varepsilon(\lambda) = \int \eta_\varepsilon \wedge \lambda$ is defined and $T_\varepsilon(\lambda) = L(T_\varepsilon)$. For $\varepsilon \to 0$, T_ε converges to T in the flat norm, and, since L is continuous w.r.t. $\mathbf{F}$-convergence, we have $\lim_{\varepsilon\to 0} L(T_\varepsilon) = L(T)$.

Hence, it suffices to show that $T_\varepsilon(\lambda) = T(\lambda_\varepsilon)$ holds also for the merely bounded and Lebesgue-measurable form λ.

Since the support of T is compact, we may assume that also the support of λ is compact and hence that λ is integrable w.r.t. Lebesgue measure. So we can find a sequence of smooth forms λ^i converging to λ in L^1. Let ${\lambda^i}_\varepsilon$ originate from λ^i by convolution with φ_ε. Then

$${\lambda^i}_\varepsilon(x) - \lambda_\varepsilon(x) = \int \varphi_\varepsilon(x-y)\big(\lambda^i(y) - \lambda(y)\big)\,dy$$

and hence

$$\left|\lambda^i{}_\varepsilon(x) - \lambda_\varepsilon(x)\right| \le \sup|\varphi_\varepsilon| \int \left|\lambda^i(y) - \lambda(y)\right| dy\,.$$

Therefore, $\lambda^i{}_\varepsilon$ converges, for $i \to \infty$, uniformly to λ_ε. So for the current $T = \mu_T \llcorner \vec{T}$, we have

$$\lim_{i\to\infty} T(\lambda^i{}_\varepsilon) = \lim_{i\to\infty} \int \langle \vec{T}, \lambda^i{}_\varepsilon\rangle\, d\mu_T = \int \langle \vec{T}, \lambda_\varepsilon\rangle\, d\mu_T = T(\lambda_\varepsilon)\,.$$

On the other hand, since λ^i is smooth, we have $T(\lambda^i{}_\varepsilon) = T_\varepsilon(\lambda^i)$, and since T_ε is smooth, L^1-convergence of $\lambda^i \to \lambda$ yields

$$\lim_{i\to\infty} T(\lambda^i{}_\varepsilon) = \lim_{i\to\infty} T_\varepsilon(\lambda^i) = T_\varepsilon(\lambda)\,.$$

So $T(\lambda_\varepsilon) = T_\varepsilon(\lambda) = L(T_\varepsilon)$ and $\lim\limits_{\varepsilon\to 0} T(\lambda_\varepsilon) = \lim\limits_{\varepsilon\to 0} L(T_\varepsilon) = L(T)$. $\square$

Using Lemma 3.4 and Lemma 3.7 we can easily prove (4):

3.8 Theorem. *On a compact and oriented Riemannian manifold M the comass norm $\mathcal{S}^*$ on $H^m_{\mathrm{dR}}(M)$ is dual to the stable norm $\mathcal{S}$ on $H_m(M,\mathbb{R})$ with respect to the natural pairing between homology and cohomology.*

Remark. This statement follows from [Fe2, 4.10]. The proof given here elaborates the one sketched in [Gr, 4.35].

Proof. If $\alpha \in H_m(M,\mathbb{R})$ is represented by $T \in \mathbf{N}_m(M)$ and if $\beta \in H^m_{\mathrm{dR}}(M)$ is represented by $\omega \in \Omega^m M$ then

$$[\alpha,\beta] = T(\omega) \le \mathbf{M}(T)\|\omega\|_\infty\,.$$

This implies

$$[\alpha,\beta] \le \mathcal{S}(\alpha)\,\mathcal{S}^*(\beta)\,.$$

It remains to show that

$$\mathcal{S}(\alpha) \le \sup_{\mathcal{S}^*(\beta)\le 1} [\alpha,\beta]\,. \tag{6}$$

According to Proposition 2.4 we can choose a minimizing $T \in \mathbf{N}_m(M)$ in α and according to Lemma 3.4 there exists a calibration L calibrating T. Then

$$L(T) = \mathbf{M}(T) = \mathcal{S}(\alpha)\,.$$

By convolution we mollify a measurable m-form λ representing L to obtain closed forms $\lambda_\varepsilon \in \Omega^m M$ such that

$$\lim_{\varepsilon\to 0} \|\lambda_\varepsilon\|_\infty = \operatorname*{ess\,sup}_{x\in M} \|\lambda(x)\| = 1\,.$$

Now Lemma 3.7 implies

$$\mathcal{S}(\alpha) = L(T) = \lim_{\varepsilon\to 0} T(\lambda_\varepsilon) = \lim_{\varepsilon\to 0}[\alpha, \beta_\varepsilon]$$

where $\beta_\varepsilon = [\lambda_\varepsilon] \in H^m_{\mathrm{dR}}(M)$ satisfies

$$\liminf_{\varepsilon\to 0} \mathcal{S}^*(\beta_\varepsilon) \le \lim_{\varepsilon\to 0} \|\lambda_\varepsilon\|_\infty = 1\,.$$

This proves (6). $\square$

4 Calibrations and Minimizing Currents in Codimension One

In this section we will show that the singular set of a codimension one calibration L contains the singular sets of all closed integer multiplicity rectifiable currents calibrated by L. For the definition of the set $\mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ of integer multiplicity rectifiable currents, for the definition of the regular part $\operatorname{reg} T$ of a homologically minimizing $T \in \mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ and for the regularity theory for such currents we refer to [Si, §27 and §37]. Here, we note that if $T \in \mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ and if ∂T has locally finite mass, then $T \in \mathbf{N}^{\mathrm{loc}}_{n-1}(M)$. Moreover, if $T \in \mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ is closed and homologically minimizing then $\operatorname{reg} T$ is a smooth hypersurface in M oriented by a smooth unit $(n-1)$-vector field $\vec{T}$ and

$$T(\omega) = \int_{\operatorname{reg} T} \omega$$

for all $\omega \in \Omega^{n-1}_0 M$.

4.1 Lemma. *Suppose that the flat cocycle $L \in \mathbf{N}_{n-1}(M)^*$ is a calibration and that $T \in \mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ is closed and calibrated by L.*

Then, for every $p \in \operatorname{reg} T \cap \operatorname{reg} L$, we have $D_L(p) = \vec{T}(p)^\flat$.

Here, $\vec{T}(p)^\flat \in \Lambda^{n-1} T_p M$ denotes the $(n-1)$-covector satisfying $\langle \xi, \vec{T}(p)^\flat\rangle = g_p(\xi, \vec{T}(p))$ for all $(n-1)$-vectors $\xi \in \Lambda_{n-1} T_p M$. For the definition of $\vec{T}$ see Section 2.2.

Proof. Let $p \in \operatorname{reg} T \cap \operatorname{reg} L$. Since the statement is local we may work in a local chart, i.e. in $\mathbb{R}^n$ equipped with a Riemannian metric g. Mass and volume will be defined w.r.t. g. To distinguish metric terms which refer to g from the Euclidean ones, we will mark them by a subscript or superscript g. We may assume that p is the origin, that the metric g_p at the origin coincides with the standard Euclidean scalar product and that $\operatorname{reg} T$ is a hyperplane P through the origin, oriented by the unit $(n-1)$-vector $\xi := \vec{T}(p)$.

For any $(n-1)$-simplex σ in P with the orientation of T, we have $L\big([\![\sigma]\!]\big) = \mathbf{M}_g\big([\![\sigma]\!]\big) = \operatorname{vol}_g(\sigma)$. Therefore, every sequence σ_i of $(n-1)$-simplices with $p \in \sigma_i$ and $\xi(\sigma_i) = \xi$ for every $i \in \mathbb{N}$, and $\lim_{i\to\infty} \operatorname{diam}(\sigma_i) = 0$ satisfies

$$\lim_{i\to\infty} \frac{L\big([\![\sigma]\!]_i\big)}{\operatorname{vol}(\sigma_i)} = \lim_{i\to\infty} \frac{\operatorname{vol}_g(\sigma_i)}{\operatorname{vol}(\sigma_i)} = 1\,.$$

Recalling Definition 3.5 we get $\langle \xi, D_L(p)\rangle = 1$. Since $|D_L(p)| = \|D_L(p)\| \le 1$ and since $|\xi| = 1$, this implies that $D_L(p) = \xi^\flat = \vec{T}(p)^\flat$. The crucial point is that the comass norm on the space of $(n-1)$-covectors is Euclidean and hence strictly convex. $\square$

4.2 Theorem. *Suppose that the flat cocycle $L \in \mathbf{N}_{n-1}(M)^*$ is a calibration and that $T \in \mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ is closed and calibrated by L. Then* $\operatorname{sing} T \subseteq \operatorname{sing} L$.

In particular, the union of the singular sets of all closed $T \in \mathcal{R}^{\mathrm{loc}}_{n-1}(M)$ calibrated by L is a Lebesgue null set.

Proof. Note that T is homologically mass-minimizing. Assume that $p \in \operatorname{spt} T$ is a regular point of L. We have to show that $p \in \operatorname{reg} T$. As above, since the statement is of local nature, we may work in local coordinates. By the Decomposition Theorem for codimension one rectifiable currents, cf. [Si, 27.6], we may assume that T is of multiplicity one. We identify some neighborhood of p in M with $\mathbb{R}^n$, equipped with a Riemannian metric g, the point p with the origin 0, such that $g_p = g_0$ corresponds to the standard scalar product of $\mathbb{R}^n$. Mass w.r.t. g is denoted by $\mathbf{M}_g$ while all other metric terms refer to the Euclidean metric.

The regularity theory for mass-minimizing currents implies that there exists a tangent cone C of T at p, i.e., there exists a sequence $r_i > 0$ converging to zero such that the sequence $T_i := \mu_{1/r_i\#} T$ converges weakly to a closed multiplicity one current $C \in \mathcal{R}^{\mathrm{loc}}_{n-1}(\mathbb{R}^n)$ (cf. [Si, Theorem 37.4]). This current C is a cone, i.e., $\mu_{r\#} C = C$ for every $r > 0$. The tangent cone C is mass-minimizing with respect to the metric g_0. Hence C is given by the smooth

hypersurface $\operatorname{reg} C$ (which is a cone, too) with a possible singular set of dimension $\leq n-8$.

We will show that C is calibrated by the cochain L_0 represented by the constant $(n-1)$-form $\lambda_0(x) := D_L(p)$. From this we will conclude that $\operatorname{spt} C$ is a hyperplane, and then regularity theory shows that p is a regular point of T.

Let $W \subseteq \mathbb{R}^n$ be an open set with compact closure that contains the origin such that $\tilde{C} := C \llcorner W \in \mathbf{N}_{n-1}(\mathbb{R}^n)$. By the arguments used in the proofs of [Si, Theorem 37.2] and [Fe1, 5.4.2], there exists a sequence of compact sets $K_i \subseteq \mathbb{R}^n$ such that the currents $S_i := T_i \llcorner K_i \in \mathbf{N}_{n-1}(\mathbb{R}^n)$ satisfy

$$\lim_{i\to\infty} \mathbf{F}(S_i - \tilde{C}) = 0 \quad \text{and} \quad \lim_{i\to\infty} \mathbf{M}_{g_i}(S_i) = \mathbf{M}(\tilde{C}) \,,$$

where $g_i := \frac{1}{r_i^2}(\mu_{r_i})^* g$. Note that $S_i = (\mu_{1/r_i})_\#(T \llcorner \tilde{K}_i)$ where $\tilde{K}_i := \mu_{r_i} K_i$.

If we set $L_i := \frac{1}{r_i^{n-1}}(\mu_{r_i})^\# L$, Proposition 3.6 (b) gives us $\lim_{i\to\infty} L_i(S) = L_0(S)$ for every $S \in \mathbf{N}_{n-1}(\mathbb{R}^n)$. By Proposition 3.6 (a) the sequence of cochains L_i is uniformly bounded w.r.t. the flat norm. Hence

$$\lim_{i\to\infty} L_i(S_i) = L_0(\tilde{C}) \,,$$

where L_0 is given by the constant $(n-1)$-form $\lambda_0(x) = D_L(p)$. Since L_0 is closed and $\mathbf{F}(L_0) = \|D_L(p)\| \leq 1$, L_0 is a calibration.

Using the fact that L calibrates T, we get

$$\begin{aligned} L_i(S_i) &= \frac{1}{r_i^{n-1}} L\big((\mu_{r_i})_\# S_i\big) = \frac{1}{r_i^{n-1}} L\big((\mu_{r_i})_\#(\mu_{1/r_i})_\#(T \llcorner \tilde{K}_i)\big) \\ &= \frac{1}{r_i^{n-1}} L(T \llcorner \tilde{K}_i) = \frac{1}{r_i^{n-1}} \mathbf{M}(T \llcorner \tilde{K}_i) = \mathbf{M}_{g_i}(S_i) \,. \end{aligned}$$

Therefore, $L_0(\tilde{C}) = \lim_{i\to\infty} L_i(S_i) = \lim_{i\to\infty} \mathbf{M}_{g_i}(S_i) = \mathbf{M}(\tilde{C})$, i.e., L_0 calibrates $\tilde{C}$.

Now, Lemma 4.1 implies that $\vec{C}(x)^\flat = D_L(p)$ for every $x \in \operatorname{reg} C \cap W$. In particular, $\vec{C}(x)$ does not depend on $x \in \operatorname{reg} C$. Since $\partial C = 0$, we can conclude that $\operatorname{spt} C$ is a hyperplane. Since C is of multiplicity one this implies that the density of C at p, and hence the density of T at p, is one, and it follows from [Fe1, 5.4.6] together with [Fe1, 5.4.5 (2)] that $p \in \operatorname{reg} T$. This proves the first statement.

The second one follows from the fact that, by [Wh, IX, Theorem 5A], $\operatorname{sing} L$ is a Lebesgue null set. $\square$

5 The Structure of Homologically Minimizing Closed Currents in Codimension One

5.1 The case of boundaries

It is not difficult to prove that $\mathbf{N}_n^{\mathrm{loc}}(M)$ is precisely the set of currents of the form $T_f = [\![M]\!] \llcorner f$, where $f \in BV^{\mathrm{loc}}(M)$ is of locally bounded variation, cf. [Fe1, 4.5.7], and

$$T_f(\omega) = \int_M f\,\omega$$

for all $\omega \in \Omega_0^n M$. Hence a codimension one boundary $T \in \mathbf{N}_{n-1}^{\mathrm{loc}}(M)$ is given as $T = \partial T_f$ for some $f \in BV^{\mathrm{loc}}(M)$. Additionally, we will assume that T is homologically minimizing, cf. Definition 2.3. We will give an overview over results on the structure of such T, which are proved in [Fe1, 4.5.9] and [AB1], [AB2].

The BV-function f can be chosen to be (upper or lower) semicontinuous. For $s \in \mathbb{R}$ consider the sets $\{x \in M \mid f(x) > s\}$ and $\{x \in M \mid f(x) \geq s\}$ and let $T_{s+} := \partial[\![\{x \in M \mid f(x) > s\}]\!]$ and $T_{s-} := \partial[\![\{x \in M \mid f(x) \geq s\}]\!]$. Here, if $A \subseteq M$ is measurable then $[\![A]\!] \in (\Omega_0^n M)^*$ denotes the current defined by $[\![A]\!](\omega) = \int_A \omega$ for $\omega \in \Omega_0^n M$. We have:

1. $T_{s+} = T_{s-}$ for all but countably many $s \in \mathbb{R}$.
2. $T_{s+}, T_{s-} \in \mathcal{R}_{n-1}^{\mathrm{loc}}(M)$ for every $s \in \mathbb{R}$.
3. Every T_{s+}, T_{s-} is homologically minimizing, so
4. for each $T_{s\pm}$, its regular part $\operatorname{reg} T_{s\pm}$ is a smooth hypersurface, dense in $\operatorname{spt} T_{s\pm}$, and the singular part $\operatorname{sing} T_{s\pm} = \operatorname{spt} T_{s\pm} \setminus \operatorname{reg} T_{s\pm}$ is of Hausdorff dimension at most $n-8$.
5. $T = \int_{\mathbb{R}} T_{s+}\,ds = \int_{\mathbb{R}} T_{s-}\,ds$.
6. $\mathbf{M}_g(T) = \int_{\mathbb{R}} \mathbf{M}_g(T_{s+})\,ds = \int_{\mathbb{R}} \mathbf{M}_g(T_{s-})\,ds$.
7. $\operatorname{spt} T = \bigcup_{s\in\mathbb{R}} \big(\operatorname{spt} T_{s+} \cup \operatorname{spt} T_{s-}\big)$.
8. For every $s \in \mathbb{R}$,

$$T_{s+} = \mathbf{F}\text{-}\lim_{h\to 0+} \frac{1}{h}\int_s^{s+h} T_{t\pm}\,dt \quad \text{and} \quad T_{s-} = \mathbf{F}\text{-}\lim_{h\to 0+} \frac{1}{h}\int_{s-h}^{s} T_{t\pm}\,dt\,.$$

Let $J = \{s \in \mathbb{R} \mid T_{s-} \neq T_{s+}\}$.

5.1 Definition. For $s \in J$ let G_s denote the interior of $\{x \in M \mid f(x) = s\}$. The sets G_s are called the *gaps* of T.

Then we have

$$M \setminus \operatorname{spt} T = \bigcup_{s \in J} G_s \,.$$

For the boundary of G_s in the sense of currents, we have $\partial[\![G_s]\!] = T_{s-} - T_{s+}$. The set theoretic boundary ∂G of G_s consists of connected components of $\operatorname{spt} T_{s-} \cup \operatorname{spt} T_{s+}$. Outside the singular sets $\operatorname{sing} T_{s-} \cup \operatorname{sing} T_{s+}$ this boundary is smooth and consists of connected components of the hypersurfaces $\operatorname{reg} T_{s-}$ and $\operatorname{reg} T_{s+}$. We denote the smooth part of ∂G by $\operatorname{reg} \partial G$.

We call $\operatorname{reg} T := \bigcup_{s\in\mathbb{R}} \big(\operatorname{reg} T_{s+} \cup \operatorname{reg} T_{s-}\big)$ the *regular* set of T and $\operatorname{sing} T := \bigcup_{s\in\mathbb{R}} \big(\operatorname{sing} T_{s+} \cup \operatorname{sing} T_{s-}\big) = \operatorname{spt} T \setminus \operatorname{reg} T$ the *singular* set of T. Although the regularity theory for mass-minimizing rectifiable $(n-1)$-currents implies that the singular set of each $T_{s\pm}$ is of Hausdorff dimension $\leq n-8$, *a priori* it is not clear whether their union $\operatorname{sing} T$ is small. However, the following lemma implies that it is at least a Lebesgue null set, cf. Corollary 5.3.

5.2 Lemma. *Suppose $T \in \mathbf{N}^{\mathrm{loc}}_{n-1}(M)$ is a boundary calibrated by the flat cocycle L. Then:*

(a) For every $p \in \operatorname{reg} T \cap \operatorname{reg} L$, the $(n-1)$-covector $D_L(p)$ is uniquely determined by T, $D_L(p) = \vec{T}_{s\pm}(p)^\flat$ if $p \in \operatorname{reg} T_{s\pm}$.

(b) $\operatorname{sing} T \subseteq \operatorname{sing} L$. In particular, $\operatorname{sing} T$ is a Lebesgue null set.

Proof. Since T is calibrated by L, it is homologically mass-minimizing. So we can apply the list of statements above. Points 6 and 8 imply that L calibrates each T_{s+} and T_{s-}, cf. Lemma 3.2.

Then, by Lemma 4.1, we have $D_L(p) = \vec{T}_{s\pm}(p)^\flat$ for every $p \in \operatorname{reg} T_{s\pm} \cap \operatorname{reg} L$. This proves (a). Statement (b) follows immediately from Theorem 4.2. $\square$

5.3 Corollary. *Suppose $T \in \mathbf{N}^{\mathrm{loc}}_{n-1}(M)$ is a locally mass-minimizing boundary. Then $\operatorname{sing} T$ (as defined above) is a Lebesgue null set.*

Proof. Since the statement is of local nature we may assume that T has compact support. Using the Hahn–Banach Theorem like in Section 3.2 we get a calibration that calibrates T (cf. [Fe2, 4.10]). Then the statement follows from Lemma 5.2. $\square$

5.2 The covering space M_α associated to $\alpha \in H_{n-1}(M, \mathbb{R})$

From now on we assume that M is compact. In order to apply the results of the preceding subsection to closed currents that do not bound, we will lift these to an appropriate (infinite) covering space of M. Here we present the relevant material from topology.

Using currents, we can describe the *Poincaré duality* isomorphism between $H^1_{\mathrm{dR}}(M)$ and $H_{n-1}(M, \mathbb{R})$ as follows (cf. [dR, Theorem 14]). Every $\alpha \in H_{n-1}(M, \mathbb{R})$ can be represented by a smooth closed current T_η, where $\eta \in \Omega^1 M$ is closed. Then the cohomology class of η depends only on α and is called the *Poincaré dual* $\alpha^{\mathrm{PD}} \in H^1_{\mathrm{dR}}(M)$ of α. The natural pairing between $H_1(M, \mathbb{R})$ and $H^1_{\mathrm{dR}}(M)$ yields the *intersection form*

$$I \colon H_1(M, \mathbb{R}) \times H_{n-1}(M, \mathbb{R}) \to \mathbb{R}, \ I(h, \alpha) = [h, \alpha^{\mathrm{PD}}] \,.$$

Explicitly, if $\alpha = [T_\eta]$ and h is represented by a real Lipschitz 1-cycle c, then

$$I(h, \alpha) = \int_c \eta \,. \tag{7}$$

The image of the Hurewicz homomorphism $H \colon \pi_1(M) \to H_1(M, \mathbb{R})$ is the set of all integer classes in $H_1(M, \mathbb{R})$ and will be denoted by $H_1(M, \mathbb{Z})_{\mathbb{R}}$. With respect to the intersection form I the lattice $H_1(M, \mathbb{Z})_{\mathbb{R}}$ in $H_1(M, \mathbb{R})$ is dual to the lattice $H_{n-1}(M, \mathbb{Z})$ in $H_{n-1}(M, \mathbb{R})$.

We set

$$K(\alpha) = \big\{ k \in H_1(M, \mathbb{Z})_{\mathbb{R}} \bigm| I(k, \alpha) = 0 \big\} \tag{8}$$

and $\tilde{K}(\alpha) = H^{-1}\big(K(\alpha)\big) \subseteq \pi_1(M)$. Now the covering $p \colon M_\alpha \to M$ associated to $\alpha \in H_{n-1}(M, \mathbb{R})$ is given by

$$M_\alpha = \tilde{M} / \tilde{K}(\alpha) = \bar{M} / K(\alpha) \,,$$

where $\tilde{M}$ denotes the universal covering and $\bar{M} = \tilde{M} / \ker H$ denotes the Abelian covering of M. Equations (7) and (8) imply that $p \colon M_\alpha \to M$ is the smallest covering space of M such that $p^*\eta$ is exact. The group of deck transformations of p is isomorphic to

$$H_1(M, \mathbb{Z})_{\mathbb{R}} / K(\alpha) \simeq \pi_1(M) / \tilde{K}(\alpha) \simeq \mathbb{Z}^{b - \operatorname{rk} K(\alpha)} \tag{9}$$

where $b = b_1(M) = \dim H_1(M, \mathbb{R})$ is the first Betti number of M. We denote the deck transformation corresponding to $k \in H_1(M, \mathbb{Z})_{\mathbb{R}}$ by $\tau_k \colon M_\alpha \to M_\alpha$. Note that $\tau_k \neq \mathrm{id}_{M_\alpha}$ iff $I(k, \alpha) \neq 0$.

Recall that in the introduction we defined $V(\alpha)$ to be the smallest linear subspace of $H_{n-1}(M,\mathbb{R})$ that is spanned by integer classes and contains α. From the preceding discussion one can conclude that $V(\alpha)$ is the orthogonal complement of $K(\alpha)$ with respect to I,

$$V(\alpha) = \left\{\beta \in H_{n-1}(M,\mathbb{R}) \;\middle|\; I(k,\beta) = 0 \text{ for all } k \in K(\alpha)\right\}.$$

If $\beta \in V(\alpha)$ then obviously $V(\beta) \subseteq V(\alpha)$. Hence the preceding equation implies that $K(\alpha) \subseteq K(\beta)$ and $\tilde{K}(\alpha) \subseteq \tilde{K}(\beta)$. This proves:

(10) If $\beta \in V(\alpha)$ and if $\eta \in \Omega^1 M$ represents $\beta^{\mathrm{PD}} \in H^1_{\mathrm{dR}}(M)$, then $p^*\eta \in \Omega^1 M_\alpha$ is exact.

Now let $\eta \in \Omega^1 M$ represent $\alpha^{\mathrm{PD}} \in H^1_{\mathrm{dR}}(M)$. Then there exists a primitive $g \in C^\infty(M_\alpha,\mathbb{R})$ of $p^*\eta$, i.e., $dg = p^*\eta$, and (7) implies

$$g(\tau_k x) = g(x) + I(k,\alpha) \tag{11}$$

for all $k \in H_1(M,\mathbb{Z})_{\mathbb{R}}$ and all $x \in M_\alpha$.

5.3 The lift to M_α

Let $\alpha \in H_{n-1}(M,\mathbb{R})$ and consider the covering $p\colon M_\alpha \to M$ with the induced metric on M_α. The *lift* $p^\# T \in \mathbf{N}^{\mathrm{loc}}_{n-1}(M_\alpha)$ of $T \in \mathbf{N}_{n-1}(M)$ to M_α is defined by

$$\big(p^\# T\big)(\omega) = \big(((p|U)^{-1})_\# T\big)(\omega)$$

provided that $\omega \in \Omega^{n-1}M_\alpha$ has compact support in an open subset $U \subseteq M_\alpha$ on which p is injective. Note that $\mathrm{spt}(p^\# T) = p^{-1}(\mathrm{spt}\, T)$. For $\eta \in \Omega^1 M$ we have $p^\#(T_\eta) = T_{p^*\eta}$. If $p^*\eta$ is exact, $p^*\eta = dg$, then $p^\#(T_\eta)$ is a boundary

$$p^\#(T_\eta) = -\partial T_g\,,$$

cf. Section 2.2, Example 2.

5.4 Lemma. *Suppose $T \in \mathbf{N}_{n-1}(M)$ is a closed normal current representing $\alpha \in H_{n-1}(M,\mathbb{R})$. Then there exists $f \in BV^{\mathrm{loc}}(M_\alpha,\mathbb{R})$ such that*

$$p^\# T = \partial T_f$$

and

$$f \circ \tau_k = f - I(k,\alpha) \tag{12}$$

for every $k \in H_1(M,\mathbb{Z})_{\mathbb{R}}$.

Proof. According to Section 5.2, α can also be represented by a current of the form T_η, $\eta \in \Omega^1 M$ a closed smooth 1-form. Therefore $T = T_\eta + \partial S$ for some $S \in \mathbf{N}_n(M)$. S is of the form $S = T_h$ for some $h \in BV(M)$, cf. Section 5.1. So, if $p^*\eta = dg$ for $g \in C^\infty(M_\alpha, \mathbb{R})$, then the lift $p^\# T$ of T to M_α has the form

$$p^\# T = p^\#(T_\eta) + \partial(p^\# S) = -\partial T_g + \partial T_{h \circ p} = \partial T_f\,,$$

where $f := -g + h \circ p \in BV^{\mathrm{loc}}(M_\alpha)$. Since g satisfies (11), f satisfies (12). $\square$

5.5 Lemma. *Suppose that $T \in \mathbf{N}_{n-1}(M)$ is a closed normal current, that $L \in \mathbf{N}_{n-1}(M)^*$ is a calibration, represented by the bounded measurable $(n-1)$-form λ, and that L calibrates T.*

Then the lift $p^\# L$ of L, defined by $(p^\# L)(S) = L(p_\# S)$ for every $S \in \mathbf{N}_{n-1}(M_\alpha)$, is a calibration represented by $p^*\lambda$ and calibrates the lift $\bar{T} := p^\# T$ of T and every leaf $\bar{T}_s$ of $\bar{T}$.

Proof. Since p is distance-nonincreasing we have $\mathbf{F}(p_\# S) \le \mathbf{F}(S)$ for every $S \in \mathbf{N}_{n-1}(M_\alpha)$. This shows that $\mathbf{F}(p^\# L) \le 1$. Then the statements follow from Section 3.3 and Lemma 3.2, and from Points 6 and 8 in Section 5.1. $\square$

In particular, if T is a minimizer in α, then its lift $\bar{T} = p^\# T$ to M_α is homologically minimizing. Therefore it has the properties described in Section 5.1 above. In particular, $M_\alpha \setminus \operatorname{spt} \bar{T}$ is the countable union of gaps.

5.6 Corollary. *For each gap G of $\bar{T}$ the restriction $p|G$ of p to G is injective. In particular, G has finite volume.*

Proof. By Definition 5.1, $G = G_s$ for some $s \in J \subseteq \mathbb{R}$. If $\tau_k \ne \mathrm{id}_{M_\alpha}$ is a deck transformation of $p\colon M_\alpha \to M$ then $I(k, \alpha) \ne 0$, and (12) implies that $\tau_k G_s \cap G_s = \emptyset$. Hence, $p|G_s$ is injective and $\mathrm{vol}_n(G) \le \mathrm{vol}_n(M) < \infty$. $\square$

6 Proof of Theorem 1.1

It suffices to consider $\alpha \in H_{n-1}(M, \mathbb{R}) \setminus \{0\}$. To show that the restriction $\mathcal{S}|V(\alpha)$ of $\mathcal{S}$ to $V(\alpha)$ is differentiable at α, we have to prove the following. If l_1, l_2 are two subderivatives of $\mathcal{S}$ at α and if $\beta \in V(\alpha)$, then $l_1(\beta) = l_2(\beta)$. Let $T \in \mathbf{N}_{n-1}(M)$ be a minimizer in α, i.e., $[T] = \alpha$ and $\mathcal{S}(\alpha) = \mathbf{M}(T)$. By Lemma 3.4 there are flat cochains L_1 and L_2 corresponding to l_1 and l_2,

respectively, that calibrate T, i.e., $L_i(S) \le \mathbf{M}(S)$ for every $S \in \mathbf{N}_{n-1}(M)$ and $L_i(T) = \mathbf{M}(T)$, for $i = 1, 2$. By Section 3.3 there are Lebesgue-measurable $(n-1)$-forms $\lambda_1 := D_{L_1}$, $\lambda_2 := D_{L_2}$ representing the cochains L_1, L_2 with $|\lambda_1| \le 1$, $|\lambda_2| \le 1$.

We represent β by a smooth current T_η, $\eta \in \Omega^1 M$ a smooth closed 1-form with $[\eta] = \beta^{\mathrm{PD}} \in H^1_{\mathrm{dR}}(M)$. Then $l_i(\beta) = L_i(T_\eta) = \int_M \eta \wedge \lambda_i$ for $i = 1, 2$. Using Lemma 5.5 and applying Lemma 5.2 to $p^\# L$ and $p^\# T$ we conclude that $\lambda_1 = \lambda_2$ Lebesgue almost everywhere on $\operatorname{spt} T$. Hence it is enough to show that

$$\int_{M \setminus \operatorname{spt} T} \eta \wedge (\lambda_1 - \lambda_2) = 0\,.$$

Consider the covering $p\colon M_\alpha \to M$ associated to α and let $\bar{T} := p^\# T$ be the lift of T to M_α. Then $M \setminus \operatorname{spt} T = p(M_\alpha \setminus \operatorname{spt} \bar{T})$, where $M_\alpha \setminus \operatorname{spt} \bar{T} = \bigcup_{s \in J} G_s$ is the union of at most countably many gaps, cf. Section 5. By Corollary 5.6, for every gap G, the restriction $p|G\colon G \to M$ of p to G is injective. Hence it suffices to show that

$$\int_G p^*\big(\eta \wedge (\lambda_1 - \lambda_2)\big) = 0$$

for every gap G.

Since $\beta \in V(\alpha)$, the 1-form $p^*\eta$ is exact, cf. (10). So there exists $g \in C^\infty(M_\alpha, \mathbb{R})$ such that $p^*\eta = dg$ and we have to show that

$$\int_G dg \wedge (\bar{\lambda}_1 - \bar{\lambda}_2) = 0\,,$$

where $\bar{\lambda}_i = p^*\lambda_i$.

Our aim is to cut off the "ends" of G and apply Stokes's Theorem to the remaining compact domain. For any given $\delta > 0$, we will show that

$$\left| \int_G dg \wedge \omega \right| < \delta\,,$$

where $\omega := \bar{\lambda}_1 - \bar{\lambda}_2$. Choose $x_0 \in G$ and let $\tilde{d}$ be a smooth function such that $\sup_{x \in M_\alpha} \big|\tilde{d}(x) - d(x_0, x)\big| \le 1$ and $\operatorname{Lip}(\tilde{d}) \le 2$. For $r > 0$ set $B_r := \{x \in M_\alpha \mid \tilde{d}(x) < r\}$. Since $p^*\eta = dg$ is bounded, there exists $r_0 > 0$ such that $\sup_{x \in B_r} |g(x)| \le 2Cr$ for all $r \ge r_0$, where $C = \sup_{M_\alpha} |dg|$.

By Corollary 5.6, $\operatorname{vol}_n(G)$ is finite. By the coarea formula, we have

$$\int_0^\infty \operatorname{vol}_{n-1}(G \cap \partial B_r)\, dr \le \operatorname{Lip}(\tilde{d}) \operatorname{vol}_n(G) < \infty\,.$$

Therefore we can find $r > r_0$ such that ∂B_r is a smooth hypersurface which meets the regular part $\operatorname{reg}\partial G$ of ∂G transversely and such that

$$\operatorname{vol}_n(G \smallsetminus B_r) < \frac{\delta}{8C} \text{ and } \operatorname{vol}_{n-1}(G \cap \partial B_r) < \frac{\delta}{24Cr}\,.$$

In particular, since $|\bar{\lambda}_i| \le 1$, we have

$$\left| \int_{G \smallsetminus B_r} dg \wedge \omega \right| < \frac{\delta}{4}\,.$$

Since the $(n-1)$-forms $\bar{\lambda}_1$, $\bar{\lambda}_2$ are not smooth, we have to pass to smooth forms in order to apply Stokes's Theorem. We do this by mollifying the λ_i by means of convolution. Thus we find smooth approximations λ_i^ε of the $\bar{\lambda}_i$ which satisfy $\sup|\lambda_i^\varepsilon| \le \frac{3}{2}$ and

$$\left| \int_{G \cap B_r} dg \wedge (\omega^\varepsilon - \omega) \right| < \frac{\delta}{4}\,,$$

where $\omega^\varepsilon = \lambda_1^\varepsilon - \lambda_2^\varepsilon$. Since, for $i = 1, 2$, L_i is a cocycle, $\bar{\lambda}_i$ is weakly closed and hence $d\lambda_i^\varepsilon = 0$. Now, the boundary currents $T^+ := T_{s+} \llcorner \partial G_s$ and $T^- := T_{s-} \llcorner \partial G_s$ of $G = G_s$ are calibrated by the lift $\bar{L}_i = p^\# L_i$, represented by $\bar{\lambda}_i$, cf. Lemma 5.5. By Lemma 3.2 (c) we have

$$\bar{L}_i\big((T^\pm \llcorner g) \llcorner B_r\big) = \int_{\operatorname{reg} T^\pm \cap B_r} g \; d\operatorname{vol}_{n-1}\,.$$

In particular, $\bar{L}_1\big((T^\pm \llcorner g) \llcorner B_r\big) = \bar{L}_2\big((T^\pm \llcorner g) \llcorner B_r\big)$. Hence Lemma 3.7 implies

$$\begin{aligned} &\lim_{\varepsilon \to 0} \left(\int_{\operatorname{reg} T^\pm \cap B_r} g\lambda_1^\varepsilon - \int_{\operatorname{reg} T^\pm \cap B_r} g\lambda_2^\varepsilon \right) = \\ &\bar{L}_1\big((T^\pm \llcorner g) \llcorner B_r\big) - \bar{L}_2\big((T^\pm \llcorner g) \llcorner B_r\big) = 0\,. \end{aligned}$$

So we can choose ε such that

$$\left| \int_{\partial G \cap B_r} g\, \omega^\varepsilon \right| < \frac{\delta}{4}\,.$$

The inequalities $\sup_{B_r}|g| \le 2Cr$, $\sup|\omega^\varepsilon| \le 3$ and $\operatorname{vol}_{n-1}(G \cap \partial B_r) < \frac{\delta}{24Cr}$ imply

$$\left| \int_{G \cap \partial B_r} g\, \omega^\varepsilon \right| < \frac{\delta}{4}\,.$$

Now, since $d\omega^\varepsilon = 0$, we have $d(g\omega^\varepsilon) = dg \wedge \omega^\varepsilon$. Since the boundary of $G \cap B_r$ is a smooth hypersurface except for a set of zero $(n-1)$-dimensional Hausdorff measure, we can apply Stokes's Theorem (see e.g. [Si, 14.3]) to get

$$\int_{G \cap B_r} dg \wedge \omega^\varepsilon = \int_{\operatorname{reg} \partial G \cap B_r} g\, \omega^\varepsilon + \int_{G \cap \partial B_r} g\, \omega^\varepsilon .$$

This implies

$$\left| \int_{G \cap B_r} dg \wedge \omega^\varepsilon \right| \le \left| \int_{\operatorname{reg} \partial G \cap B_r} g\, \omega^\varepsilon \right| + \left| \int_{G \cap \partial B_r} g\, \omega^\varepsilon \right| < \frac{\delta}{2}$$

and

$$\left| \int_G dg \wedge \omega \right| \le \left| \int_{G \cap B_r} dg \wedge \omega^\varepsilon \right| + \left| \int_{G \cap B_r} dg \wedge (\omega^\varepsilon - \omega) \right| + \left| \int_{G \setminus B_r} dg \wedge \omega \right| < \delta .$$

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